1) In a plane triangle, find the maximum & minimum value of $\cos A \cos B \cdot \cos C$.

$$A+B+C=\pi$$

$$C = \pi-(A+B)$$

$$\cos C = \cos(\pi-(A+B)) = -\cos(A+B)$$

$$\Rightarrow f = -\cos A \cos B \cos(A+B)$$

$$\frac{\partial f}{\partial A} = -\cos B\left[-\sin A\cos(A+B) + \cos A(-\sin(A+B))\right]$$

$$\cos B \sin(2A+B) = 0$$

$$\frac{\partial f}{\partial B} = -\cos A\,(\sin(A+2B)) = 0$$

only possible if $A = B = \pi/3$

$$\frac{\partial^2 f}{\partial A^2} = r = 2\cos B\cos(2A+B)$$

$$\frac{\partial^2 f}{\partial B^2} = t = +2\cos A\cos(A+2B)$$

$$\frac{\partial^2 f}{\partial B\,\partial A} = s = -\sin B\sin(2A+B)\,\cancel{\cos(2A+2B)} + \cos B\cos(2A+B)$$

for $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $r = -1$

$s = -1/2 \qquad t = -1$

$rt - s^2 \Rightarrow 1-\frac{1}{4} > 0, \quad r<0$

maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right) \Rightarrow C = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

maximum value $\Rightarrow \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$

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2) Find the maximum & minimum values of $\sin x \sin y \sin(x+y)$.

$f = \sin x \sin y \sin(x+y)$

$\frac{\partial f}{\partial y} = \sin x \sin(x+2y) = 0 \quad -①$

$\frac{\partial f}{\partial x} = \sin y \cdot \sin(2x+y) = 0 \quad -②$

from ① & ②,

if, $x = y = \pi/3$ or $x = y = \frac{2\pi}{3}$

$\frac{\partial^2 f}{\partial x^2} = 2\sin y \cos(2x+y) = r$

$\frac{\partial^2 f}{\partial y \partial x} = \cos y \sin(2x+y) + \sin y \cos(2x+y) - \sin(2x+2y) = s$

$\frac{\partial^2 f}{\partial y^2} = 2\sin x \cos(x+2y) = t$

at $(\pi/3, \pi/3)$, $r = -\sqrt{3}$, $s = -\sqrt{3}/2$, $t = -\sqrt{3}$ [Maxima]

at $(\frac{2\pi}{3}, \frac{2\pi}{3})$, $r = \sqrt{3}$, $s = \sqrt{3}/2$, $t = \sqrt{3}$ [Minima]

$\sin x \cdot \sin y \cdot \sin(x+y)$

maximum value $\Rightarrow$ $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$

minimum value $\Rightarrow$ $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \left(\frac{-\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{8}$

3) Find the dimensions of the box open at the top of maximum capacity whose surface is 432 sq.cm

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$F(x,y,z) \Rightarrow F = \underbrace{xyz}_{\text{maximum}} - \lambda(xy + 2xz + 2yz - 432) = 0$

$\frac{\partial F}{\partial x} = yz - \lambda y - 2\lambda y = 0$ — (1)

$\frac{\partial F}{\partial y} = xz - \lambda x - 2\lambda z = 0$ — (2)

$\frac{\partial F}{\partial z} = xy - 2\lambda x - 2\lambda y = 0$ — (3)

$x(1) - y(2) \Rightarrow \quad -2\lambda z(x-y) = 0$

$z = 0$ (not possible), so $\Rightarrow x = y$

$(2) \times y - (3) \times z \Rightarrow \quad -\lambda x[y - 2z] = 0 \Rightarrow \quad y = 2z$

$x = y, \quad y = 2z$

$4z^2 + 4z^2 + 4z^2 = 432.$

$z = \pm 6 \Rightarrow \underline{\underline{+6}}$

$x = y = 12$

Length = Breadth = 12 cm & height = 6 cm

4) Find the maximum & minimum distance from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$

$F(x,y) \Rightarrow \sqrt{x^2+y^2} - \lambda(5x^2 + 6xy + 5y^2 - 8) = 0$

$\frac{\partial F}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} - \lambda 10x - 6\lambda y = 0$ — (1)

$\frac{\partial F}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} - \lambda 10y - 6\lambda x = 0$ — (2)

$y \times (1) - (2)x \Rightarrow \quad -6\lambda(y^2 - x^2) = 0 \Rightarrow \quad y^2 = x^2$

$\underline{y = \pm x}$

If $5x^2 + 6xy + 5y^2 = 8$

$y = +x \Rightarrow 5x^2 + 6x^2 + 5x^2 = 8 \Rightarrow x^2 = 1/2 \Rightarrow \boxed{x = 1/\sqrt{2}}$

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If $y = -x$, $5x^2 - 6x^2 + 5x^2 = 8$

$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Maximum dist. $\rightarrow \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$

minimum dist. $\Rightarrow \sqrt{2+2} = 2$

5) Divide 24 into 3 parts such that the continued product of the first, square of second & cube of the third may be maximum.

let 2 parts be $x$ & $y$

3rd $\rightarrow z = 24 - (x+y)$

$f(x,y) \Rightarrow [24-(x+y)]x^2 \cdot y^3$

$\log f = 2\log x + 3\log y + \log(24-(x+y))$

partially differentiate w.r.t $x$

$\frac{1}{f} f_x \Rightarrow \frac{2}{x} + \frac{1 \cdot (-1)}{24-(x+y)}$

similarly,

$f_y = f\left[\frac{2}{x} - \frac{1}{24-x-y}\right] = 0 \rightarrow f_x = f\left(\frac{2}{x} + \frac{1(-1)}{24-(x+y)}\right)$

$f_{xx} \Rightarrow f_x\left(\frac{2}{x} - \frac{1}{24-x-y}\right) + f\left[\frac{-2}{x^2} - \frac{1}{(24-x-y)^2}\right]$

$f_{yy} \Rightarrow f_y\left[\frac{3}{y} - \frac{1}{24-x-y}\right] + f\left[\frac{-3}{y^2} - \frac{1}{(24-x-y)^2}\right]$

$f_{xy} \Rightarrow f_y\left(\frac{2}{x} - \frac{1}{24-x-y}\right) + f\left[\frac{-1}{(24-x-y)^2}\right]$

$\Rightarrow f_x = 0$, $\frac{2}{x} = \frac{1}{24-x-y} \Rightarrow 48 - 3x - 2y = 0$ — (1)

$f_y = 0$, $\Rightarrow 72 - 3x - 4y = 0$ — (2)

$\boxed{y = 12}$ $\boxed{x = 8}$ $\boxed{z = 4}$

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first term $= z = 4$

second term $= x = 8$

third term $= y = 12$

$$f_{xx} = f_x\left(\frac{2}{x} - \frac{1}{24-x-y}\right) + f\left(\frac{-2}{x^2} - \frac{1}{(24-x-y)^2}\right)$$

$\boxed{f(x) = 0}$

$$\Rightarrow f\left(-\frac{2}{x^2} - \frac{1}{(24-x-y)^2}\right) = \frac{3f}{32}$$

$$f_{yy} \Rightarrow f\left[\frac{-3}{144} - \frac{1}{16}\right] = -\frac{f}{12}$$

$$f_{xy} = \frac{-f}{16}$$

$rt - s^2 \Rightarrow$ +ve

& $f_{xx} < 0$, so

$f(x,y)$ is maximum at $x = 8$, $y = 12$, $z = 4$

6) By successive differentiation of

$\int_0^1 x^m dx = \frac{1}{m+1}$ w.r.t $m$, evaluate $\int_0^1 x^m (\log x)^n dx$

$$\left(\int_0^1 x^m dx\right)' \Rightarrow \left(\int_0^1 x^m \cdot \log x \cdot dx\right)' = \int_0^1 x^m (\log x)^2$$

$\int_0^1 x^m (\log x)^m dx \rightarrow$ differentiating $n$ times w.r.t $x$

$$\left(\frac{1}{m+1}\right)' \Rightarrow \left(\frac{-1}{(m+1)^2}\right)' \Rightarrow \frac{2}{(m+1)^3} \Rightarrow \frac{-2.3}{(m+1)^4}$$

$$\frac{(-1)^n \, n!}{(m+1)^{n+1}}$$

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7) Prove that $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx = \tan^{-1}\left(\frac{1}{a}\right)$.

Hence show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

$$F(a) = \int_0^\infty e^{-ax} \frac{\sin x}{x} dx = \tan^{-1}\left(\frac{1}{a}\right)$$

$$F'(a) = \int_0^\infty \frac{\sin x}{x} \left(e^{-ax}(-x)\right) = -\int_0^\infty e^{-ax} . \sin x$$

$$\left\{ \because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a\sin bx - b\cos bx) \right\}$$

$$F'(a) = -\frac{e^{-ax}}{a^2+1} (-a\sin x - \cos x)$$

$$\Rightarrow \frac{e^{-ax}}{1+a^2} (a\sin x + \cos x)_0^\infty \Rightarrow 0 - \frac{1}{1+a^2}$$

$$F'(a) = \frac{-1}{1+a^2}$$

integrating on both sides

$$F(a) = -\int \frac{1}{1+a^2} da = \cot^{-1} a + c = \tan^{-1}\left(\frac{1}{a}\right) + c$$

$$F(\infty) = \tan^{-1}\left(\frac{1}{\infty}\right) + c \Rightarrow 0 + c = F(\infty)$$

$$= \int_0^\infty \underbrace{e^{-a\infty}}_{\to 0} . \frac{\sin \infty}{\infty} d(\infty)$$

$$\boxed{c = 0}$$

$$F(a) = \tan^{-1}\left(\frac{1}{a}\right)$$

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} dx = \tan^{-1}\left(\frac{1}{a}\right)$$

$$\tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

putting $a = 0 \Rightarrow \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

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8) ~~Prove that $\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx = \tan^{-1}\left(\frac{1}{a}\right)$.~~
~~Hence show that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$~~

8) Prove that $\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}\,dx = \pi\sin^{-1}\alpha$

diff. w.r.t $\alpha \Rightarrow \int_0^\pi \frac{1}{\cos x}\cdot\frac{\cos x}{(1+\alpha\cos x)}\,dx$

$$\Rightarrow \int_0^\pi \frac{1}{1+\alpha\cos x}\,\cancel{dx} \quad -(1)$$

~~put~~ $\tan\left(\frac{x}{2}\right) = t \Rightarrow \sec^2\left(\frac{x}{2}\right)\cdot\frac{1}{2}\,dx = dt$

$$\int_0^\pi \frac{1}{1+\cos x}\,dx \Rightarrow \int_0^\pi \frac{1}{1+\alpha\left[\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right]}$$

$$\Rightarrow \int_0^\pi \frac{1}{1+\tan^2(x/2)+\alpha-\alpha\tan^2 x/2}$$

$$\int_0^\pi \frac{\sec^2 x/2}{1+\tan^2 x/2+\alpha-\alpha\tan^2(x/2)}\,dx$$

$\tan(x/2) = t$ , $\frac{\sec^2(x/2)}{2}\,dx = dt$

$$2\int_0^\pi \frac{dt}{(1+\alpha)+t^2-\alpha t^2} = 2\int_0^\pi \frac{dt}{(1+\alpha)+(1-\alpha)t^2}$$

$$= \frac{2}{1-\alpha}\int_0^\pi \frac{dt}{\frac{(1+\alpha)}{(1-\alpha)}+t^2}$$

$$\frac{2}{1-\alpha}\int_0^\pi \frac{dt}{t^2+\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\right)^2} \quad \left[\because \int\frac{dt}{t^2+a^2} \Rightarrow \frac{1}{a}\tan^{-1}\left(\frac{t}{a}\right)\right]$$

$$\frac{2}{1-\alpha}\left[\frac{1}{\sqrt{\frac{1+\alpha}{1-\alpha}}}\tan^{-1}\left(\frac{t}{\sqrt{\frac{1+\alpha}{1-\alpha}}}\right)\right]_0^\pi$$

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$$\frac{2}{1-\alpha}\sqrt{\frac{1-\alpha}{1+\alpha}}\left(\tan^{-1}\sqrt{\frac{1-\alpha}{1+\alpha}}\cdot\tan\left(\frac{x}{2}\right)\right)\Big|_0^{\pi}$$

$$\left[\int_0^{\pi}\frac{1}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}} \quad (a>b)\right]$$

from (1), $\frac{d\phi}{da} \Rightarrow \int_0^{\pi}\frac{1}{a+b\cos x} = \frac{\pi}{\sqrt{1-a^2}}$

$$d\phi = \frac{\pi}{\sqrt{1-a^2}}\,da \quad -(2)$$

integrate both sides,

$$\phi = \int\frac{\pi}{\sqrt{1-a^2}}\,da$$

$$\phi = \pi\sin^{-1}a + c \quad -(3)$$

in (3) if $a=0$, $\phi = c$

$$\phi = \int_0^{\pi}\frac{\log(1+a\cos x)}{\cos x}\,dx$$

if $a=0$, $\int_0^{\pi}\frac{\log x}{\cos x}\,dx = 0$

~~so, $\phi = 0$~~, ~~so~~, $c = 0$

$$\underline{\phi = \pi\sin^{-1}a}$$

Hence proved

9) If $y = \int_0^{x} f(t)\sin[k(x-t)]\,dt$, prove that $y$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2y = k\,f(x)$$

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$$y = \int_0^x f(t) \cdot \sin(k(x-t))\, dt$$

$$\frac{dy}{dx} = \int_0^x f(t) \cos(k(x-t)) \cdot k\, dt + f(x) \times 0$$

$$\frac{d^2y}{dx^2} \Rightarrow -k^2 \int_0^x f(t) \sin k(x-t)\, dt + f(x) \cdot k$$

$$\Rightarrow -k^2 y + f(x) \cdot k$$

$$\frac{d^2y}{dx^2} = -k^2 y + f(x) \cdot k$$

$$\frac{d^2y}{dx^2} + k^2 y = f(x) \cdot k$$

Hence proved.

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